Additional Comments and Conclusions

Presented has been an idea for possibly improving structural efficiency using fiber-reinforced materials. Specifically, the notion of using a curvilinear fiber format in flat plates with central circular holes has been studied. The paper is not all encompassing, and it was not intended to be that way. This note simply suggests that the curvilinear format has the potential for using reinforcing fibers more effectively. Designs using the curvilinear format have been contrasted with conventional straight-line design counterparts. Results have been compared with a standard quasi-isotropic design. Tensile and compressive buckling loads have been studied. It can be concluded that in tension the curvilinear designs studied lead to improved performance. In compression, the buckling loads are not as high as they are for quasi-isotropic laminates, but the buckling loads for the curvilinear design are no lower than the buckling loads for their straight-line non-quasi-isotropic counterparts.

Before closing, a comment should be made regarding manufacturing. As with many designs, the curvilinear configurations discussed here may present some manufacturing problems. For example, the $(\pm 45/C_6)_S$ design calls for the fiber angles along line BB′ to not be horizontal. When considering a complete plate, rather than just one quarter, this is a problem. For a complete plate, the fibers along line BB′ must be horizontal. Along line BB′ there cannot be a slightly negative angle specified by the top quarter of the plate and a slightly positive angle specified by the bottom quarter. In practice, the fiber angles in the elements along line BB′ must be adjusted to make them horizontal. This may not impact the increased efficiency. Sensitivity analyses, to determine which areas of the plate are least effected by deviations of the fiber angle from the ideal, can be conducted. It is suspected that the load capacity is not particularly sensitive to fiber angle along line BB′. Thus, the fiber angles can be adjusted to achieve manufacturing compatibility and not significantly effect performance.

Also regarding manufacturing, Knoblach[11] recently investigated the use of electric fields to align short chopped fibers in specific directions while the material was being manufactured. There were problems with the electrodes and fiber-fiber interaction but the concept worked. Plates were produced that had better stiffness and strength properties than plates with random fiber orientation, i.e., quasi-isotropic.

More work needs to be done. Studies are needed to pursue laminates with other combinations of straight and curvilinear layers. Currently, studies are underway to investigate the use of the curvilinear format to increase the buckling resistance. Manufacturing issues are also being addressed. The character of postbuckling performance and failure with he curvilinear format also deserves attention. These developments will be reported on at a later date as work progresses.

Acknowledgments

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Analytical Model Improvement Using Measured Modes and Submatrices

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Introduction

THERE are numerous methods available in the literature describing how to improve an analytical model of a linear structure in order to match the experimentally identified mode shapes and frequencies. These methods are characterized by the assumptions and goals made with respect to three sets of data: an analytical mass matrix, an analytical stiffness matrix, and a set of measured modes. Assuming that the analytical mass matrix is correct, methods for adjusting the measured modes to achieve orthogonality have been proposed e.g., Ref. 1. The opposite approach, which has been also taken, assumes that the measured modes are correct and adjusts the analytical mass matrix.[2] When the orthogonality condition between the analytical mass matrix and the measured modes is satisfied, various methods can be employed to correct the analytical stiffness matrix.[1-4]

This Note addresses the procedure of correcting the analytical mass and stiffness matrices simultaneously to match the measured modes instead of updating the mass matrix first and the stiffness matrix subsequently. In order to achieve this, the previously proposed submatrix approach to the analytical stiffness matrix adjustment[5] is extended while preserving the characteristics of the method. Thus, the connectivity and the consistency of the stiffness matrix are preserved and the mass orthogonalized mode shapes are not required. A computationally efficient pseudoinverse solution is employed to solve the resulting system of linear equations for the submatrix scaling factors that are used to correct the mass and stiffness matrices. The method also incorporates a capability of reducing finite

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element mass and stiffness matrices to test degrees-of-freedom (DOF) for a direct analysis/test correlation.

Development of the Method

The eigenvalue problem governing the dynamic characteristics of an undamped n DOF linear system is represented by

$$[K_n][\Phi_n] = [M_n][\Phi_n][\omega_n^2] \tag{1}$$

where $[M_n]$ and $[K_n]$ are the system mass and stiffness matrices, respectively; $[\Phi_n]$ is the system modal matrix of dimension $n \times l$; and $[\omega_n^2]$ is a diagonal matrix containing l system eigenvalues.

In this Note, in order to perform the adjustment of the mass and stiffness matrices to match the measured mode data, the stiffness and mass matrices are expanded into a linear sum of submatrices such as

$$[K_n] = [K_a] + \sum_{j=1}^{p} a_j[K_j] \tag{2}$$

$$[M_n] = [M_a] + \sum_{j=1}^{q} b_j[M_j] \tag{3}$$

where $[K_a]$ and $[M_a]$ are the analytical stiffness and mass matrices, respectively; $[K_j]$ and $[M_j]$ are the jth stiffness and mass submatrices transformed into the global coordinates, respectively; a_j and b_j are the jth scaling factors for the stiffness and mass submatrices, respectively; and p and q are the total number of stiffness and mass submatrices, respectively. The submatrices represent a single element or a group of elements of the structure having the same assumed geometry, material properties, boundary conditions, and modeling assumptions. Mass and stiffness submatrices are, in general, linear combinations of the element mass and stiffness matrices in the global coordinate system, respectively. The correction of the analytical matrices is performed by adjusting the submatrix scaling factors. A significant reduction of parameters can be achieved by judiciously grouping the structural elements with the same characteristics. An illustration of the concept of submatrices and scaling factors is available in Ref. 5.

In general, the system mass and stiffness matrices in Eq. (1), often generated by the finite element method, possess a large number of DOF in comparison to the number of sensors or test DOF used in a modal survey. To allow a direct comparison between the finite element modes and the measured modes, the finite element mass and stiffness matrices are often reduced to the test DOF. The reduced representation is referred to as a test analysis model (TAM). The modal matrix in Eq. (1) is partitioned into two complementary sets, $[\Phi_{na}]$ and $[\Phi_{nd}]$, such that

$$[\Phi_n] = [B]\begin{bmatrix}[\Phi_{na}]\\ [\Phi_{nd}]\end{bmatrix} = [B][\Phi_n'] \tag{4}$$

where $[\Phi_{na}]$ contains r active or independent DOF that are to be retained in the TAM and $[\Phi_{nd}]$ contains the dependent DOF to be eliminated from the representation. Substitute Eqs. (2–4) into Eq. (1) and premultiply by $[B]^T$ to obtain

$$\left([K_{aB}] + \sum_{j=1}^{p} a_j[K_{jB}]\right)[\Phi_n'] = \left([M_{aB}] + \sum_{j=1}^{q} b_j[M_{jB}]\right)[\Phi_n'][\omega_n^2] \tag{5}$$

where

$$[K_{aB}] = [B]^T[K_a][B]$$
$$[K_{jB}] = [B]^T[K_j][B]$$
$$[M_{aB}] = [B]^T[M_a][B]$$
$$[M_{jB}] = [B]^T[M_j][B]$$

The transformation in Eq. (5) reorganizes the coefficients in the mass and stiffness matrices and the submatrices according to the DOF in $[\Phi_n']$. The Guyan reduction[6] is employed to reduce Eq. (5). Define a transformation

$$[\Phi_n'] = [T][\Phi_{na}] = \begin{bmatrix}[I_{aa}]\\ -[K_{aB}^{dd}]^{-1}[K_{aB}^{da}]\end{bmatrix}[\Phi_{na}] \tag{6}$$

where $[I_{aa}]$ is an $r \times r$ identity matrix and $[K_{aB}^{da}]$ and $[K_{aB}^{dd}]$ are the lower left and lower right partitions of the matrix $[K_{aB}]$ according to the DOF in $[\Phi_{na}]$ and $[\Phi_{nd}]$, respectively. Substitution of Eq. (6) into Eq. (5) and premultiplication by $[T]^T$ result in the reduced representation

$$\left([K_{aR}] + \sum_{j=1}^{p} a_j[K_{jR}]\right)[\Phi_{na}] = \left([M_{aR}] + \sum_{j=1}^{q} b_j[M_{jR}]\right)[\Phi_{na}][\Omega_n^2] \tag{7}$$

where

$$[K_{aR}] = [T]^T[K_{aB}][T]$$
$$[K_{jR}] = [T]^T[K_{jB}][T]$$
$$[M_{aR}] = [T]^T[M_{aB}][T]$$
$$[M_{jR}] = [T]^T[M_{jB}][T]$$

The measured modal matrix $[\Phi_m]$, and the measured frequencies $[\omega_m^2]$ are now substituted into Eq. (7) and used for the stiffness and mass matrix correction. Note that a set of nonzero scaling factors exists that makes both the left and right side of Eq. (7) zero if $[K_{aR}]$ and $[M_{aR}]$ are linear combinations of $[K_{jR}]$ and $[M_{jR}]$, respectively. To avoid having this trivial solution, we should have at least one known stiffness element or mass element in the test structure and exclude the element from the submatrices. Rearrange Eq. (7) to obtain

$$\sum_{j=1}^{p} a_j[A_j] - \sum_{j=1}^{q} b_j[B_j] = [R] \tag{8}$$

where

$$[A_j] = [K_{jR}][\Phi_m] \tag{9}$$

$$[B_j] = [M_{jR}][\Phi_m][\omega_m^2] \tag{10}$$

$$[R] = [M_{aR}][\Phi_m][\omega_m^2] - [K_{aR}][\Phi_m] \tag{11}$$

The size of matrices $[A_j]$, $[B_j]$, and $[R]$ is $r \times m$ and the matrices are known. The quantity m is the number of modes used for the correction. Define a vector

$$\{s\} = \{a_1\ a_2 \ldots a_p\ b_1\ b_2 \ldots b_q\}^T \tag{12}$$

whose elements are scaling factors. By equating each column of Eq. (8) and rearranging, we obtain

$$[L]\{s\} = \{r\} \tag{13}$$

where

$$[L] = \begin{bmatrix}\{A_1\}_1 & \{A_2\}_1 & \ldots & \{A_p\}_1 & \{-B_1\}_1 & \{-B_2\}_1 & \ldots & \{-B_q\}_1\\ \{A_1\}_2 & \{A_2\}_2 & \ldots & \{A_p\}_2 & \{-B_1\}_2 & \{-B_2\}_2 & \ldots & \{-B_q\}_2\\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot\\ \{A_1\}_m & \{A_2\}_m & \ldots & \{A_p\}_m & \{-B_1\}_m & \{-B_2\}_m & \ldots & \{-B_q\}_m\end{bmatrix} \tag{14}$$

$$\{r\} = [\{R\}_1^T \{R\}_2^T \ldots \{R\}_m^T]^T \tag{15}$$

The vectors $\{A_j\}_k$ ($j = 1,2,\ldots,p$; $k = 1,2,\ldots,m$) and $\{B_j\}_k$ ($j = 1,2,\ldots,q$; $k = 1,2,\ldots,m$) represent the kth columns of matrices $[A_j]$ and $[B_j]$, respectively. The vector $\{R\}_k$ is the kth column of the matrix $[R]$. The sizes of $[L]$ and $\{r\}$ are $\alpha \times \beta$ and $\alpha \times 1$, respectively, where $\alpha = rm$ and $\beta = p + q$.

The scaling factors in Eq. (13) can be computed using a pseudoinverse solution such that

$$\{s\} = [L]^+ \{r\} \tag{16}$$

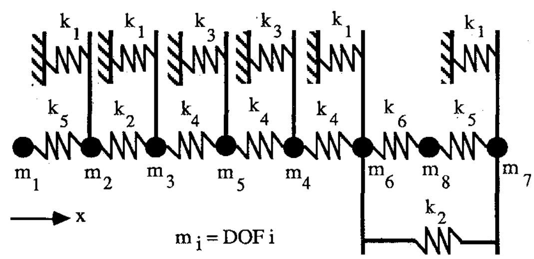
where $[L]^+$ is a pseudoinverse of the $[L]$ matrix. When $\alpha \geq \beta$ and the matrix $[L]$ is of full rank, i.e., rank $([L]) = \beta$, $[L]^+$ becomes $([L]^T[L])^{-1}[L]^T$. Even when the problem is underspecified, i.e., the row dimension of the $[L]$ matrix is smaller than the column dimension, a solution can be obtained using a minimum Euclidean norm solution. However, the solution will be less accurate in this case. The identified scaling factors are now substituted into Eq. (7) to improve the reduced analytical mass and stiffness matrices or Eqs. (2) and (3) to improve the finite element mass and stiffness matrices directly. A process of expanding the reduced mass and stiffness matrices into the finite element DOF is thus avoided.

When either the mass or the stiffness matrix is assumed or verified to be correct in addition to the measured modes, the same procedure can be used to adjust the incorrect matrix by setting the stiffness submatrix scaling factors to zero ($a_j = 0$) for the mass matrix correction and the mass submatrix scaling factors to zero ($b_j = 0$) for the stiffness matrix correction.

Numerical Example

Consider the 8 DOF mass-spring system with the exact values of masses and spring constants as shown in Fig. 1. The masses are constrained to translate in the x direction only. In this example, the mass at DOF 8 (m_8) is assumed to be known exactly and the analytical mass and stiffness matrices are to be corrected using measured mode data. The nonzero upper triangular stiffness and mass matrix coefficients of the structure in the finite element DOF are shown in Table 1. To proceed with the analytical model improvement procedure, the submatrices of Eqs. (2) and (3) should be defined. Since there are six distinct spring elements, six stiffness submatrices $[K_j]$, corresponding to the six spring constants k_j, are defined ($j = 1,2,\ldots,6$). For two distinct mass values other than the known m_8, two mass submatrices are defined: $[M_1]$ for the masses m_1, m_3, and m_4; and $[M_2]$ for the masses m_2, m_5, m_6, and m_7. The nonzero, upper triangular submatrices in global coordinates are shown in Table 2. The normalized coefficients of each submatrix represent the coefficients divided by the corresponding spring constants or mass values.

In an actual modal survey, the number of test DOF is typically less than the number of finite element DOF. In this case, the reduction of the finite element DOF to the test DOF is required for an efficient analysis/experiment correlation. In order to preserve the dynamic characteristics of the finite element model with good accuracy, 5 DOF (x_2, x_5–x_8) are selected as the test DOF for this example. The reduced representation of Eq. (7) is obtained readily using the Guyan reduction procedure described previously. The number of DOF of the TAM is



$m_1 = m_3 = m_4 = 0.05$, $m_2 = m_5 = m_6 = m_7 = 1.0$, and $m_8 = 0.5$

$k_1 = 50$, $k_2 = 40$, $k_3 = 60$, $k_4 = 20$, $k_5 = 30$, and $k_6 = 10$

Fig. 1 One-dimensional mass-spring system.

Table 1 Nonzero upper triangular stiffness and mass matrix coefficients in the finite element degree of freedom

Coefficient location	Analytical stiffness matrix	Corrected stiffness matrix	Exact stiffness matrix	Analytical mass matrix	Corrected mass matrix	Exact mass matrix
1,1	45.0	29.4	30.0	0.09	0.055	0.05
1,2	− 45.0	− 29.4	− 30.0			
2,2	135.0	118.5	120.0	0.97	1.01	1.0
2,3	− 50.0	− 38.9	− 40.0			
3,3	110.0	109.4	110.0	0.09	0.055	0.05
3,5	− 20.0	− 20.3	− 20.0			
4,4	80.0	101.0	100.0	0.09	0.055	0.05
4,5	− 20.0	− 20.3	− 20.0			
4,6	− 20.0	− 20.3	− 20.0			
5,5	80.0	101.0	100.0	0.97	1.01	1.0
6,6	125.0	119.8	120.0	0.97	1.01	1.0
6,7	− 50.0	− 38.9	− 40.0			
6,8	− 15.0	− 10.4	− 10.0			
7,7	135.0	118.5	120.0	0.97	1.01	1.0
7,8	− 45.0	− 29.4	− 30.0			
8,8	60.0	39.8	40.0	0.5	0.5	0.5

Table 2 Nonzero upper triangular coefficients of submatrices in global coordinates

$[K_1]$		$[K_2]$		$[K_3]$		$[K_4]$		$[K_5]$		$[K_6]$		$[M_1]$		$[M_2]$	
A[a]	B[b]	A	B	A	B	A	B	A	B	A	B	A	B	A	B
2,2	1.0	2,2	1.0	4,4	1.0	3,3	1.0	1,1	1.0	6,6	1.0	1,1	1.0	2,2	1.0
3,3	1.0	2,3	− 1.0			3,5	− 1.0	1,2	− 1.0	6,8	− 1.0				
				5,5	1.0							3,3	1.0	5,5	1.0
6,6	1.0	3,3	1.0			4,4	2.0	2,2	1,0	8,8	1.0	4,4	1.0	6,6	1.0
						4,5	− 1.0								
7,7	1.0	6,6	1.0			4,6	− 1.0	7,7	1.0					7,7	1.0
		6,7	− 1.0			5,5	2.0	7,8	− 1.0						
		7,7	1.0			6,6	1.0	8,8	1.0						

[a]A = Coefficient location. [b]B = Normalized coefficients.

5, i.e., $r = 5$ and the number of scaling factors is 8, i.e., $\beta = 8$. Thus, the minimum number of measured modes to have an overspecified system of linear equations is two. The measured modes are simulated by calculating the natural frequencies and the mode shapes at the test DOF using the exact mass and stiffness matrices in the finite element DOF. Three modes (modes 1, 2, and 3) are employed for the correction. The identified scaling factors are $\{s\} = \{10.20 \ -11.09 \ 20.43 \ 0.27 \ -15.62 \ -4.56 \ -0.035 \ 0.040\}^T$.
The identified scaling factors can be employed to correct the mass and stiffness matrices either in the finite element DOF using Eqs. (2) and (3) or in the test DOF using Eq. (7). The corrected stiffness and mass matrix coefficients in the finite element DOF are shown in Table 1. The mass and stiffness matrix coefficients are not exactly corrected because the measured modes in the test DOF are used for the correction and the reduced model is not exact.

Concluding Remarks

A systematic approach to improve an analytical model of a linear structure in order to match the experimentally identified mode shapes and frequencies was presented. The incorporation of the submatrix concept enables the method to preserve the connectivity and consistency of the stiffness matrix. Orthogonalization of the measured mode shapes with respect to the mass matrix is not required in the method. In order to perform a direct test/analysis correlation, the Guyan reduction technique was employed in the development of the method. The accuracy of the adjusted mass and stiffness matrices using a test analyses model depends not only on the accuracy of the measured modes but also on the accuracy of the TAM. Accurate TAM may be required to perform analytical model improvement properly. A computationally efficient pseudoinverse solution is used to solve the resulting system of linear equations for the scaling factors. If only the mass or the stiffness matrix is incorrect, the adjustment of the incorrect matrix is possible by setting the appropriate submatrix scaling factors to zero.

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Improved Approximate Methods for Computing Eigenvector Derivatives in Structural Dynamics

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Introduction

METHODS of computing eigenvector derivatives have been an active area of research since the earlier work of Fox and Kapoor.[1] Recent development in this area was surveyed by Haftka and Adleman.[2] To obtain exact solutions, Nelson's method[3] appears to be more efficient than other approaches. An improvement to the truncated modal summation representation of eigenvector derivatives was presented in a work by Wang,[4] in which a mode-acceleration type approach was used to obtain a static solution to approximate the contribution due to unavailable higher modes. The numerical performance of this method was compared with several other methods by Sutter et al.[5]

In this Note two improved approximate methods are represented. The first method is the explicit method (EM), which is the method presented in Ref. 4. Based on the same data used by the explicit method, a new implicit method (IM) is developed in this Note. In the implicit method the eigenvector derivatives are assumed to be spanned by the truncated mode shapes together with a residual static mode. The unknown coefficients are computed by a Bubnov-Galerkin method from the governing equation for eigenvector derivatives. Numerical examples show that this method drastically improves solution accuracy.

Analysis

The eigenvalue problem for undamped systems in structural dynamics is

$$K\phi = \lambda M\phi \tag{1}$$

Taking partial derivatives of Eq. (1) with respect to a design variable x yields the following governing equation for eigenvector derivatives:

$$Z_l \frac{\partial \phi_l}{\partial x} = F_l \tag{2}$$

where

$$Z_l = K - \lambda_l M \tag{3}$$

$$F_l = -\frac{\partial Z_l}{\partial x}\phi_l \tag{4}$$

eigenvalue:

$$\lambda_l = l\text{th}$$

eigenvector:

$$\phi_l = l\text{th}$$

and

$$\frac{\partial Z_l}{\partial x} = \frac{\partial K}{\partial x} - \frac{\partial \lambda_l}{\partial x} M - \lambda_l \frac{\partial M}{\partial x} \tag{5}$$

The eigenvalue derivative is given by

$$\frac{\partial \lambda_l}{\partial x} = \phi_l^T \left(\frac{\partial K}{\partial x} - \lambda_l \frac{\partial M}{\partial x}\right)\phi_l \tag{6}$$

where, for convenience, orthonormal modes are used. That is,

$$\phi_l^T M \phi_l = 1 \tag{7}$$

Exact Solution of Eigenvector Derivatives

Since the matrix Z_l is singular, the eigenvector derivatives cannot be obtained from Eq. (2) by elementary means. However, several methods for computing eigenvector derivatives have been developed in the literature. They are all derived based on Eqs. (2) and (7). For an N-degree-of-freedom system, if all the modes are available, the eigenvector derivative can be computed using the following equations[1]:

$$\frac{\partial \phi_l}{\partial x} = c_l \phi_l + \sum_{\substack{i=1 \\ i \neq l}}^{N} c_i \phi_i \tag{8}$$

where

$$c_l = -\frac{1}{2}\phi_l^T \frac{\partial M}{\partial x}\phi_l \tag{9}$$

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